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A General Theorem on Hypoellipticity

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INTRODUCTION

In this paper we introduce a notion of hypoellipticity of a differential operator with respect to two other ones and give necessary and sufficient conditions for this to occur. This notion includes all known types of hypoellipticity, such as the ordinary hypoellipticity and partial hypoellipticity (see [5 and 7]), almost hypoellipticity in the sense of Elliott [2], and the hypoellipticity with respect to a differential operator studied by Gorin and Grušin [4]. Thus our notion of hypoellipticity exhibits all known types of hypoellipticity as special cases of a general problem.

Let Ω be a nonempty open subset of R^n and P a polynomial in R^n. Following [8] we call a distribution $T \in \mathcal{D}'(\Omega)$ strongly regular with respect to the differential operator $P(D)$ if to every open set Ω' having compact closure contained in Ω (in this case we write $\Omega' \Subset \Omega$) there exists an integer $m \geqslant 0$ depending on Ω' such that $P^k(D)T$, $k = 0, 1,\ldots$, are all of order $\leqslant m$ in Ω', i.e., the restrictions of $P^k(D)T$ to Ω' are all in $\mathcal{D}'^m(\Omega')$; here $P^k(D)$, $k = 1, 2,\ldots$, are the successive iterates of $P(D)$ and $P^0(D)T = T$. We denote by $\mathscr{E}_P(\Omega)$ the linear space of all distributions in Ω which are strongly regular with respect to $P(D)$.

Consider now a differential operator $W(D)$ (with constant coefficients) and two spaces $\mathscr{E}_P(\Omega)$, $\mathscr{E}_Q(\Omega)$ corresponding to the differential operators $P(D)$, $Q(D)$ respectively. The operator $W(D)$ is said to be (P, Q)-hypoelliptic if, for any open set $\Omega \subset R^n$, every solution $u \in \mathscr{E}_P(\Omega)$ of the equation

$$W(D)u = 0 \tag{1}$$

is in $\mathscr{E}_Q(\Omega)$.

We prove the following theorem.

THEOREM. *The differential operator $W(D)$ is (P, Q)-hypoelliptic if and only if the polynomials P, Q, and W satisfy one of the equivalent conditions*:

(I) *$Q(\zeta)$ is bounded on every set of $\zeta \in C^n$ where $W(\zeta) = 0$ and both $P(\zeta)$ and* Im *ζ are bounded*;

(II) *there are constants γ, $C > 0$ such that*

$$|Q(\zeta)|^\gamma \leqslant C(1 + |P(\zeta)|)(1 + |\operatorname{Im} \zeta|), \qquad \zeta \in C^n, \quad W(\zeta) = 0.$$

By specifying P and/or Q we obtain necessary and sufficient conditions for hypoellipticity of $W(D)$ in all cases mentioned above.

1. NECESSARY CONDITIONS

Given an open set Ω and a polynomial P in R^n, we denote by $C_P{}^\lambda(\Omega)$, where λ is an integer $\geqslant 0$, the space of all C^λ-functions f in Ω such that $P^k(D)\, D^\alpha f$, $|\alpha| \leqslant \lambda$, $k = 0, 1,\ldots$, are continuous; here $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also denote by $\tilde{\mathscr{E}}_P(\Omega)$ and $\tilde{C}_P{}^\lambda(\Omega)$ the subspaces of $\mathscr{E}_P(\Omega)$ and $C_P{}^\lambda(\Omega)$, respectively, consisting of solutions of Eq. (1).

PROPOSITION 1. *If $\tilde{\mathscr{E}}_p(\Omega) \subset \tilde{\mathscr{E}}_Q(\Omega)$, then to any open set $\Omega' \Subset \Omega$ there exists an integer $\lambda \geqslant 0$ such that the restriction mapping $f \to f \mid \Omega'$ maps $\tilde{C}_P{}^\lambda(\Omega)$ into $\tilde{C}_Q{}^0(\Omega')$.*

The proof of this proposition is analogous to the proof of Theorem 1 in [8] and we omit it.

In view of Proposition 1 it suffices to find necessary conditions for the inclusion

$$\{f \mid \Omega' : f \in \tilde{C}_P{}^\lambda(\Omega)\} \subset \tilde{C}_Q{}^0(\Omega'). \tag{2}$$

We accomplish this by means of a standard argument based on the closed graph theorem and the Seidenberg–Tarski theorem (see, e.g., [3]).

Let Ω_j, $j = 0, 1,\ldots$, be open sets such that

$$\Omega_j \Subset \Omega_{j+1}, j = 0, 1,\ldots, \qquad \text{and} \qquad \bigcup_{j=0}^{\infty} \Omega_j = \Omega. \tag{3}$$

We define the topology in $\tilde{C}_P{}^\lambda(\Omega)$ by means of the seminorms

$$v_j(f) = \sup |P^k(D)\, D^\alpha f(x)|,$$

where the supremum is taken over $x \in \Omega_j$, $|\alpha| \leqslant \lambda$ and $k \leqslant j$. Similarly, if Ω_j', $j = 0, 1,\ldots$, are open sets satisfying conditions analogous to (3) with Ω

replaced by Ω', we define the topology in $\tilde{C}_Q{}^0(\Omega')$ by means of the seminorms

$$w_j(f) = \sup_{x\in\Omega_j{}', k\leqslant j} |Q^k(D)f(x)|.$$

Then $\tilde{C}_P{}^\lambda(\Omega)$ and $\tilde{C}_Q{}^0(\Omega')$ become Frèchet spaces. The restriction mapping $\tilde{C}_P{}^\lambda(\Omega) \to \tilde{C}_Q{}^0(\Omega')$ is closed and therefore continuous, by the closed graph theorem. Thus, to each integer $l \geqslant 0$ there exists an integer $k \geqslant 0$ and a constant $C > 0$ such that

$$w_l(f) \leqslant Cv_k(f), \qquad f \in \tilde{C}_P{}^\lambda(\Omega).$$

Applying this inequality to the function

$$f(x) = e^{i\langle x,\zeta\rangle},$$

where $\zeta \in C^n$ and $W(\zeta) = 0$, we obtain

LEMMA 1. *If the inclusion* (2) *holds, then for each integer* $l \geqslant 0$ *there exists an integer* $k \geqslant 0$ *and constants* $C, c > 0$ *such that*

$$|Q^l(\zeta)| \leqslant C(1 + |\xi|^\lambda)(1 + |P^k(\zeta)|)\, e^{c|\eta|}, \qquad W(\zeta) = 0,$$

where $\zeta = \xi + i\eta \in C^n$ *and* $\xi, \eta \in R^n$.

Proof of Necessity of (I)

Denote by $N(P)$, $N(P, a)$, R_a and I_a the sets of all $\zeta = \xi + i\eta \in C^n$ such that $P(\zeta) = 0$, $|P(\zeta)| \leqslant a$, $|\xi| \leqslant a$ and $|\eta| \leqslant a$, respectively.

Suppose there are $a, b \geqslant 0$ such that $Q(\zeta)$ is not bounded on $M = N(W) \cap N(P, a) \cap I_b$. Then the function

$$s(t) = \sup_{\zeta\in M\cap R_t} |Q(\zeta)|$$

is defined and continuous for sufficiently large t, and

$$s(t) \to \infty \qquad \text{as} \quad t \to \infty. \tag{4}$$

But, for a given t, $s(t)$ is the largest of all s such that the equations and inequalities

$$|W(\xi + i\eta)|^2 = 0, \qquad |P(\xi + i\eta)|^2 \leqslant a^2, \qquad |\eta|^2 \leqslant b^2$$
$$|Q(\xi + i\eta)|^2 = s^2, \qquad |\xi|^2 \leqslant t^2, \qquad s \geqslant 0, \qquad t \geqslant 0,$$

have a solution $\xi, \eta \in R^n$. It follows from the Seidenberg–Tarski theorem (see [5, p. 276; or 7, p. 501]) that, for sufficiently large t, $s(t)$ is an algebraic function. From (4) we then must have that

$$s(t) > t^h$$

for some $h > 0$ and all sufficiently large t. On the other hand, $s(t)$ is assumed for some $\xi = \xi(t)$, $\eta = \eta(t)$ and $|\xi(t)| \leqslant t$. Now choosing $l > \lambda h^{-1}$ and applying Lemma 1 we obtain a contradiction. Therefore $Q(\zeta)$ is bounded on M and condition (I) follows immediately.

Proof of Equivalence of (I) *and* (II)

It is obvious that (II) implies (I); we prove the converse. The notation is the same as in the preceding proof.

Consider the real polynomial

$$H(\xi, \eta, s, t) = (s^2 - |\eta|^2 - |P(\xi + i\eta)|^2)^2 + (t^2 - |Q(\xi + i\eta)|^2)^2 + |W(\xi + i\eta)|^2$$

of $2n + 2$ real variables. By condition (I), the surface

$$H(\xi, \eta, s, t) = 0$$

is contained in a domain defined by an inequality

$$|s| > g(|t|)$$

where $g(\tau) \to \infty$ as $\tau \to \infty$. In fact, if $|Q(\zeta)| \to \infty$, $\zeta = \xi + i\eta \in N(W)$, then $|\eta| + |P(\zeta)| \to \infty$; otherwise we could find a sequence $\zeta_n \in M$, for some a and b, such that $|Q(\zeta_n)| \to \infty$, contrary to (I). It follows, by applying a theorem of Gorin [3, Theorem 4.1], that there exist constants γ, $C' > 0$ satisfying the condition

$$|Q(\zeta)|^\gamma \leqslant |\eta| + |P(\zeta)| + C'.$$

This implies (II) with some other constant C.

2. Sufficient Conditions

The proof of the sufficiency of condition (II) will be based on a suitable version of the theorem of Ehrenpreis [1] and Palamodov [6] on integral (exponential) representation of solutions of Eq. (1).

First we observe that, without loss of generality, we can make the following assumptions:

(A) the coefficient in $W(\zeta) = W(\zeta_1, \ldots, \zeta_n)$ corresponding to the highest power of ζ_1 is a constant;

(B) Ω is the open ball S_a with center at the origin and radius a.

We recall that, by the Paley–Wiener theorem, the Fourier transform $\hat{\varphi}$ of a function $\varphi \in \mathscr{D}(S_a)$ can be continued in C^n as an entire function satisfying the growth conditions

$$\| \hat{\varphi} \|_m = \sup_{\zeta \in C^n} | \hat{\varphi}(\zeta)| \, (1 + | \zeta |)^m \, e^{-b|\eta|} < \infty \tag{5}$$

$\zeta = \xi + i\eta$, $m = 0, 1, \ldots$, for some $b < a$. We denote by Z the space of all Fourier transforms of functions from $\mathscr{D}(S_a)$ and by Z^b, $0 < b < a$, the subspace of Z consisting of functions satisfying conditions (5). The topology in Z^b is defined by the system of norms $\| \; \|_m$, $m = 0, 1, \ldots$, and Z is the inductive limit of the Z^b's. Then the Fourier transformation is a topological isomorphism of $\mathscr{D}(S_a)$ onto Z.

To every distribution $u \in \mathscr{D}'(S_a)$ there corresponds a continuous linear functional $\hat{u}$ on Z ($\hat{u}$ is the Fourier transform of u) defined by the equation

$$\langle u, \check{\varphi} \rangle = \langle \hat{u}, \hat{\varphi} \rangle,$$

where $\varphi \in \mathscr{D}(S_a)$ and $\check{\varphi}(x) = \varphi(-x)$. Moreover, for every positive $b < a$ there exists an integer $m \geqslant 0$ such that $\hat{u}$ is continuous on the space Z^b equipped with the single norm $\| \; \|_m$.

Let

$$W = W_1^{r_1} \cdots W_s^{r_s}$$

be the decomposition of W into irreducible factors and let N^k be the variety of zeros of W_k, i.e. $N^k = N(W_k)$. Making use of (A) we can associate with W the multiplicity variety

$$V = \left(N^1, \text{identity}; N^1, \frac{\partial}{\partial \zeta_1}; \ldots; N^1, \frac{\partial^{r_1 - 1}}{\partial \zeta_1^{r_1 - 1}}; \ldots; N^s, \frac{\partial^{r_s - 1}}{\partial \zeta_1^{r_s - 1}}\right)$$

of "length" $r = r_1 + \cdots + r_s$ (see [1, p. 50; and 6, p. 181]). We define $Z^b(V)$ to be the space of all r-tuples $g = (g_{1,0}, \ldots, g_{1,r_1-1}, \ldots, g_{s,r_s-1})$, where $g_{k,j}$, $j = 0, \ldots, r_k - 1$, are analytic functions on N^k with the following properties:

(F_1) there exists an entire function F in C^n such that

$$\partial^j F / \partial \zeta_1{}^j = g_{k,j}, \qquad j = 0, \ldots, r_k - 1, \text{ on } N^k,$$

for all k;

(F_2) $\sup_{\zeta \in N^k} | g_{k,j}(\zeta)| (1 + | \zeta |)^m e^{-b|\eta|} < \infty, \quad m = 0, 1,\ldots,$ for all j and k.

The topology in $Z^b(V)$ is defined by means of the norms

$$\| g \|'_m = \sup | g_{k,j}(\zeta)| (1 + | \zeta |)^m e^{-b|\eta|}, \qquad m = 0, 1,\ldots,$$

where the supremum is taken over all j, k and $\zeta \in N^k$.

Suppose now that u is a solution in $\mathscr{D}'(S_a)$ of Eq. (1). Applying the methods of [1] or [6] (Fundamental Principle) one can define a continuous functional v on $Z^b(V)$ such that

$$\langle \hat{u}, \psi \rangle = \langle v, d\psi \rangle, \qquad \psi \in Z^b, \tag{6}$$

where d is the restriction map

$$d: \psi \to \psi \mid V = \left(\psi \mid N^1, \frac{\partial \psi}{\partial \zeta_1} \Big| N^1,\ldots, \frac{\partial^{r_1 - 1}\psi}{\partial \zeta_1^{r_1 - 1}} \Big| N^1,\ldots, \frac{\partial^{r_s - 1}\psi}{\partial \zeta_1^{r_s - 1}} \Big| N^s\right).$$

Also, for sufficiently large m, v remains continuous on $Z^b(V)$ endowed with the topology defined by the single norm $\| \ \|'_m$. The latter space can be isometrically imbedded into the topological product of r_1 copies of $C(N^1)$, r_2 copies of $C(N^2)$, etc., where $C(N^k)$ is the space of bounded continuous functions on N^k equipped with the sup norm. By the Hahn–Banach theorem, v can be extended to the whole product space; the extension is therefore a sum of measures. In this way one obtains for u the integral representation

$$\langle u, \check{\varphi} \rangle = \sum_{k=1}^{s} \sum_{j=0}^{r_k - 1} \int_{N^k} \frac{\partial^j \hat{\varphi}}{\partial \zeta_1{}^j} (\zeta) \frac{d\mu_j{}^k(\zeta)}{h(\zeta)}, \tag{7}$$

where $\varphi \in \mathscr{D}(S_b)$, $\mu_j{}^k$, $j = 0,\ldots, r_{k-1}$, are bounded measures with support on N^k, $k = 1,\ldots, s$, and

$$h(\zeta) = (1 + | \zeta |)^{-m} e^{b|\eta|}. \tag{8}$$

The integrals in (7) converge absolutely and uniformly for all $\hat{\varphi} \in Z^b$ such that

$$|(\partial^j \hat{\varphi} / \partial \zeta_1{}^j)(\zeta)| \leqslant M_k h(\zeta), \qquad j = 0,\ldots, r_k - 1, \quad \zeta \in N^k,$$

where M_k, $k = 1,\ldots, s$, are constants.

If $u \in \mathscr{E}_P(\Omega)$ then, for every positive $b < a$ one can find $m \geqslant 0$ such that u is continuous on $\mathscr{D}(S_b)$ endowed with any of the norms

$$||| \varphi |||_{m,l} = \sup_{\substack{0 \leqslant k \leqslant l, |\alpha| \leqslant m \\ x \in R^n}} | P^k(-D) D^\alpha \varphi(x)|, \qquad l = 0, 1,\ldots .$$

This implies, as one can easily verify, that for sufficiently large m, the functional v in (6) remains continuous on the space $Z^b(V)$ endowed with any of the norms

$$\|g\|'_{m,l} = \sup |g_{k,j}(\zeta)| (1 + |\zeta|)^m (1 + |P(\zeta)|)^{-l} e^{-b|\eta|}, \qquad l = 0, 1, \ldots .$$

Thus, proceding as before we obtain the following

LEMMA 2. *Suppose that u is a solution in $\mathscr{E}_P(S_a)$ of Eq (1) and $0 < b < a$. Then there exists an integer $m \geqslant 0$ such that, for any integer $l \geqslant 0$, we have the integral representation*

$$\langle u, \check{\varphi}\rangle = \sum_{k=1}^{s} \sum_{j=0}^{r_k-1} \int_{N^k} \frac{\partial^j \hat{\varphi}}{\partial \zeta_1{}^j}(\zeta) \frac{d\mu_j^{k,l}(\zeta)}{h_l(\zeta)} \tag{9}$$

where $\varphi \in \mathscr{D}(S_b)$, $\mu_j^{k,l}$, $j = 0, \ldots, r_k - 1$, are bounded measures with support on N^k and

$$h_l(\zeta) = (1 + |\zeta|)^{-m} (1 + |P(\zeta)|)^l e^{b|\eta|} \tag{10}$$

Remark. Note that a distribution $u \in \mathscr{D}'(S_a)$ is in $\mathscr{E}_Q(S_a)$ if it has the following property.

(RQ) For any positive $b' < a$ there exists an integer $m' \geqslant 0$ such that, for any $l' \geqslant 0$,

$$\langle Q^{l'}(D)u, \check{\varphi}\rangle = \sum_{k=1}^{s} \sum_{j=0}^{r_k-1} \int_{N^k} \frac{\partial^j \hat{\varphi}}{\partial \zeta_1{}^j}(\zeta) \frac{d\nu_j^{k,l'}(\zeta)}{h'(\zeta)}, \tag{11}$$

where $\varphi \in \mathscr{D}(S_{b'})$, $\nu_j^{k,l'}$, $j = 0, \ldots, r_k - 1$ are bounded measures with support on N^k and $h'(\zeta)$ is given by (8) with b and m replaced by b' and m', respectively.

We are now in a position to prove the sufficiency of condition (II). For $k = 1, \ldots, s$, we define

$$N_1{}^k = \{\zeta \in N^k: |Q(\zeta)|^{\gamma/2} \leqslant C(1 + |\eta|)\}$$

and

$$N_2{}^k = \{\zeta \in N^k \backslash N_1{}^k: |Q(\zeta)|^{\gamma/2} \leqslant 1 + |P(\zeta)|\},$$

where γ and C are the constants appearing in condition (II). Then N^k is a disjoint union of $N_1{}^k$ and $N_2{}^k$. In fact, if $\zeta \in N^k \backslash (N_1{}^k \cup N_2{}^k)$, then

$$|Q(\zeta)|^{\gamma/2} > C(1 + |\eta|)$$

and

$$|Q(\zeta)|^{\gamma/2} > 1 + |P(\zeta)|,$$

which is a contradiction to the inequality in (II). Therefore $N^k \backslash (N_1{}^k \cup N_2{}^k)$ is the empty set, and so $N^k = N_1{}^k \cup N_2{}^k$.

Let u be a solution in $\mathscr{E}_P(S_a)$ of Eq. (1). We show that u is a sum of two distributions having property (RQ). For this reason we fix an arbitrary positive $b' < a$ and choose b so that $b' < b < a$. By Lemma 2, u admits the representation (9)–(10), where l is any given integer $\geqslant 0$. We write

$$u = u_1 + u_2$$

where

$$\langle u_i , \check{\varphi} \rangle = \sum_{k=1}^{s} \sum_{j=0}^{r_k - 1} \int_{N_i^k} \frac{\partial^j \hat{\varphi}}{\partial \zeta_1{}^j} (\zeta) \frac{d\mu_j^{k,l}(\zeta)}{h_l(\zeta)}, \tag{12}$$

$i = 1, 2$, and the equalities hold for any $\varphi \in \mathscr{D}(S_b)$.

If q is the degree of Q and $l' \geqslant \rho = \max_{1 \leqslant k \leqslant s} r_k - 1$ then, for any $j \leqslant \rho$,

$$(\partial^j (Q^{l'} \hat{\varphi}) / \partial \zeta_1{}^j)(\zeta) = Q^{l' - \rho}(\zeta)\, d_j(\zeta, \partial/\partial\zeta_1)\, \hat{\varphi}(\zeta), \tag{13}$$

where $d_j(\zeta, \partial/\partial\zeta_1)$ is a differential operator of order $\leqslant j$ (with respect to ζ_1) whose coefficients are polynomials of degree $\leqslant \rho q$. Also, making use of (12) and (13), we obtain

$$\langle Q^{l'}(D)\, u_i , \check{\varphi} \rangle = \sum_{k=1}^{s} \sum_{j=0}^{r_k - 1} \int_{N_i^k} Q^{l' - \rho}(\zeta)\, d_j \left(\zeta, \frac{\partial}{\partial \zeta_1} \right) \hat{\varphi}(\zeta) \frac{d\mu_j^{k,l}(\zeta)}{h_l(\zeta)}, \tag{14}$$

$$i = 1, 2.$$

Now, for any $\hat{\varphi} \in Z^{b'}$, we may write

$$\frac{| Q^{l' - \rho}(\zeta)\, d_j(\zeta, \partial/\partial\zeta_1)\, \hat{\varphi}(\zeta) |}{h(\zeta)} = \frac{| Q^{l' - \rho}(\zeta) |}{e^{(b - b') |\eta|}} \frac{| d_j(\zeta, \partial/\partial\zeta_1)\, \hat{\varphi}(\zeta) |}{(1 + | \zeta |)^{\rho q}\, h'(\zeta)}, \tag{15}$$

where $h(\zeta)$ is given by (8) and $h'(\zeta) = (1 + | \zeta |)^{-m'} e^{b'|\eta|}$ with $m' = m + \rho q$. But from the definition of the $N_1{}^k$'s it follows that the first factor on the right-hand side of (15) is bounded on $\bigcup_{k=1}^{s} N_1{}^k$. Thus, defining $\nu_{1,j}^{k,l'}$ as a product of $\mu_j^{k,l}$ with a suitable bounded, continuous function on $N_1{}^k$ (for each k and j), one can express the right-hand side of (14) in the form (11), which shows that u_1 has property (RQ).

On the other hand, there exists $\lambda > 0$ such that

$$| Q(\zeta) | / (1 + | P(\zeta) |)^{\lambda} \leqslant 1, \qquad \zeta \in \bigcup_{k=1}^{s} N_2{}^k,$$

by the definition of the $N_2{}^k$'s. If $l \geqslant l'\lambda$, this again allows us to define $\nu_{2,j}^{k,l'}$ as a product of $\mu_j^{k,l}$ and a suitable bounded, continuous function on $N_2{}^k$

(for each k and j) so that the right-hand side of (14) assumes the form (11). Consequently u_2 also has property (RQ).

From the remark following Lemma 2 we conclude that $u \in \mathscr{E}_Q(S_a)$ and the necessity of condition (II) is established.

3. Remarks

We show that our (P, Q)-hypoellipticity is a generalization of the known notions of hypoellipticity mentioned in the Introduction.

Consider first the case where $P = \text{const} \neq 0$. Then $\mathscr{E}_P(\Omega) = \mathscr{D}'(\Omega)$, and so W is (P, Q)-hypoelliptic if and only if every distribution in Ω of Eq. (1) is in $\mathscr{E}_Q(\Omega)$. In this case condition (II) is necessary and sufficient for the regularity of function solutions of (1) studied by Gorin and Grušin [4]. If, in addition, $Q(D)$ is the Laplace operator $\Delta = (\partial^2/\partial x_1^2) + \cdots + (\partial^2/\partial x_n^2)$, then $\mathscr{E}_Q(\Omega) = \mathscr{E}(\Omega)$ (see [8]) and therefore the operator $W(D)$ is hypoelliptic in the ordinary sense.

Suppose now that the variables in $x = (x_1, ..., x_n)$ are split into $x' = (x_1, ..., x_t)$ and $x'' = (x_{t+1}, ..., x_n)$. We recall that a distribution $T \in \mathscr{D}'(\Omega)$ is strongly regular in x' if to every open set $\Omega' \Subset \Omega$ there exists an integer m (depending on Ω') such that, for every multiindex $\alpha' = (\alpha_1, ..., \alpha_t)$, $D^{\alpha'}T$ is of order $\leqslant m$ in Ω' (see [7, p. 453]). We denote by Δ' the Laplace operator in the variable x'.

The following proposition is probably well known but since we cannot find it in the literature, we shall give a brief outline of the proof.

PROPOSITION 2. *If $P(D) = 1 - \Delta'$ then $\mathscr{E}_P(\Omega)$ is the space of distributions in Ω strongly regular in x'.*

Proof. It is clear that every distribution in Ω strongly regular in x' belongs to $\mathscr{E}_P(\Omega)$. We prove the converse, i.e. that every distribution from $\mathscr{E}_P(\Omega)$ is strongly regular in x'.

For $k = 1, 2, ...$, let F_k be the continuous function in R^n whose Fourier transform is

$$\hat{F}_k(\xi) = (1 + |\xi'|^2)^{-k} (1 + |\xi|^2)^{-n},$$

where $\xi' = (\xi_1, ..., \xi_t)$. Then the following properties are easily verified.

(a) For every k,

$$E_k = (1 - \Delta)^n F_k$$

is a fundamental solution for $P^k(D)$, i.e.,

$$P^k(D) E_k = \delta.$$

(b) $P^j(D)F_k = F_{k-j}$, for $j = 1,..., k-1$.

(c) For every multiindex $\alpha' = (\alpha_1, ..., \alpha_t)$ and every k, $D^{\alpha'}F_k$ is a continuous function in $R^n \backslash \{0\}$.

(d) For every $l \geqslant 0$ there exists a k such that $\{D^{\alpha'}F_k : |\alpha'| \leqslant l\}$ are continuous functions in R^n.

Let l be a given (arbitrarily large) positive integer and let Ω' be any open set such that $\Omega' \Subset \Omega$. If $T \in \mathscr{E}_P(\Omega)$, we can apply the same argument as in [8] (see the proof of Theorem 3) to show that the distributions $\{D^{\alpha'}T: |\alpha'| \leqslant l\}$ are in Ω' of order $\leqslant m$, say, where m is independent of l. This proves that T is strongly regular in x'.

We now set $P(D) = 1 - \Delta'$; then $\mathscr{E}_P(\Omega)$ is the space of distributions in Ω strongly regular in x', by Proposition 2. If $Q(D) = \Delta$, then $\mathscr{E}_Q(\Omega) = \mathscr{E}(\Omega)$ and the (P, Q)-hypoellipticity of W is just the partial hypoellipticity in x'' (see [7, p. 456]).

On the other hand, if P is a constant $\neq 0$ and $Q(D) = 1 - \Delta'$, then $\mathscr{E}_P(\Omega) = \mathscr{D}'(\Omega)$ and $\mathscr{E}_Q(\Omega)$ is the space of distributions in Ω strongly regular in x'. This particular case of (P, Q)-hypoellipticity is the almost hypoellipticity in x' investigated by Elliott [2].

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